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THERMOCONVECTIVE WAVES IN A COMPRESSIBLE FLUID

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Abstract—The conditions for the propagation of one-dimensional thermoconvective waves are shown to contradict the stability criterion for a vertically unbounded fluid. This leads to the problem of two-dimensional wave propagation in a stratified fluid layer confined between horizontal walls. The eigenvalue-problem is solved by a two-parameter expansion that is singular with respect to one of the parameters. The results show that the horizontal walls do not prevent the existence of weakly damped thermoconvective waves provided that the Rayleigh-number is large. Finally the interaction of thermoconvective waves and sound waves is investigated, and the amplitudes of the various wave modes are determined.

NOMENCLATURE

b, height of the layer;
c, isentropic speed of sound;
c_p, c_v, specific heat capacities at constant pressure and constant volume, respectively;
e, specific internal energy;
$\mathbf{g}$, gravitational acceleration, $\mathbf{g} = (0, -g)$;
K, isothermal compressibility;
k, complex wave number, $k = k_r + ik_i$;
N, buoyancy frequency;
Pr, Prandtl-number;
p, pressure;
s, specific entropy;
T, temperature;
t, time;
U, amplitude of u-perturbations;
u, velocity component in x-direction;
V, amplitude of v-perturbations;
v, velocity component in y-direction;
x, horizontal coordinate;
y, vertical coordinate.

Greek symbols

β, thermal expansivity;
γ_0, parameter, cf. equations (13) and (18);
$\delta, \varepsilon, \eta$, perturbation parameters, cf. equations (40), (25), and (69), respectively;
θ, amplitude of the dimensionless temperature perturbation;
κ, thermal diffusivity;
λ, thermal conductivity;
μ, viscosity;
μ_b, bulk viscosity;
ν, kinematic viscosity;
ρ, density;
Φ, dissipation;
ω, angular frequency.

Subscripts, superscripts

0, at dimensionless quantities: first order expansion; at non-dimensionless quantities: undisturbed state;
$'$, at dimensionless quantities: differentiation with respect to the argument; at non-dimensionless quantities: perturbation quantity;
$\bar{\ }$, dimensionless quantity;
x, y, t, partial derivatives with respect to x, y, t, respectively.

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1. INTRODUCTION

THERMOCONVECTIVE waves are coupled thermal and shearing waves in a stratified fluid in the gravity field. These waves are of particular interest because they can be weakly damped despite the fact that the transport of wave energy is accomplished by means of viscosity and thermal conductivity of the fluid. The propagation process is stimulated by buoyancy forces due to the anisothermal stratification of the undisturbed basic state.

Thermoconvective waves were studied for the first time by Luikov and Berkovsky in [1], where the following problem was considered (Fig. 1): A viscous, heat conducting fluid occupies the semi-space $x > 0$. A negative temperature gradient parallel to

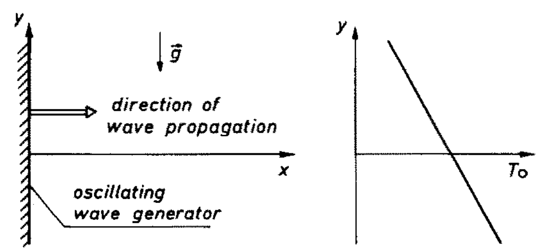

FIG. 1. Model for the propagation of one-dimensional thermoconvective waves, $T_0(y)$: temperature distribution in the undisturbed state, $\mathbf{g}$: vector of gravitational acceleration.

the vector of gravitational acceleration **g**, i.e. $\mathrm{d}T_0/\mathrm{d}y < 0$, is given. At the wall ($x = 0$) periodic oscillations with small amplitudes are prescribed, either in the form of periodic changes of the wall temperature or as an oscillatory motion of the wall. The perturbations propagate horizontally in the form of plane harmonic waves.

If the frequency of the oscillations does not exceed a certain upper limit one obtains two coupled waves (the so-called thermoconvective waves) whose properties differ remarkably from those of ordinary thermal waves and ordinary shearing waves. Table 1 shows a comparison of the complex wave numbers in terms of a small parameter ε that will be defined in Section 3. In the case of ordinary thermal and shearing waves, generated by temperature perturbations or transverse velocity perturbations in an unstratified fluid, the real and the imaginary parts of the complex wave number are equal to each other, cf.

Table 1. Comparison of complex wave numbers

Ordinary thermal wave and shearing wave:

$$k = \sqrt{\frac{\omega}{2\kappa}}\,(1+i)$$

$$k = \sqrt{\frac{\omega}{2\nu}}\,(1+i)$$

Thermoconvective waves:

$$\bar{k}_{\mathrm{I}} = 1 + iO(\varepsilon)$$

$$\bar{k}_{\mathrm{II}} = i + O(\varepsilon)$$

$$\text{with } \varepsilon \ll 1$$

e.g. [2]. Hence these waves are strongly damped over distances of the order of the wavelength. In contrast to this well known behaviour of the classical waves, the damping constant of one of the two thermoconvective waves is very small compared to the real wave number. It follows that the amplitude of this wave changes but very little over a wavelength.

Furthermore, according to a result obtained in [1], the phase angle between the thermal and transverse oscillations which is to be prescribed at the wall $x = 0$ can be chosen such that the energy introduced at the boundary is solely transported in the weakly damped thermoconvective wave. This aspect seems to be of importance for practical applications.

The first investigations on thermoconvective waves have already been extended in several aspects. Magnetohydrodynamical effects and ferromagnetic properties of the fluid have been considered [3], and the propagation of thermoconvective waves in viscoelastic media has been studied [4].

The compressibility of the fluid, however, has been neglected so far in the publications on thermoconvective waves. Of course, the compressibility of the fluid gives rise to a sound wave that propagates simultaneously with the thermoconvective waves. The coupling of these waves will be studied in Section 5 in a straightforward manner. But there is also another, more subtle compressibility effect, and this effect makes the existence of weakly damped thermoconvective waves appear questionable. Taking into account the compressibility with respect to the undisturbed, stratified state, we shall find out that the conditions for stability of the basic state and for the existence of thermoconvective waves cannot be simultaneously satisfied in a vertically unbounded fluid. Hence the fluid has to be confined between horizontal walls, and the question arises, how the properties of the waves are changed due to the horizontal walls and whether weakly damped waves are possible at all under these conditions. By means of perturbation methods it will be shown in Section 4 that the influence of the horizontal walls on the damping of thermoconvective waves is, rather surprisingly, extremely weak.

First attempts to describe the propagation of thermoconvective waves in a bounded layer have already been made by Berkovsky and Sinitsyn [5, 6]. In [5] the case of free boundaries with slip-conditions on both sides of the layer was considered. A numerical treatment of the propagation of temperature perturbations between rigid walls at subcritical and supercritical Rayleigh-numbers has been described in [6]. This problem has also been investigated experimentally [7]. Recently, these investigations have been extended by Berkovsky *et al.*† also to wave-like disturbances of developed periodic convection. In our analysis, however, thermoconvective waves are understood as small amplitude waves propagating in a stratified fluid being at rest in the undisturbed state.

2. GOVERNING EQUATIONS

The conservation equations for mass, momentum and energy are used in the following form (see e.g. [8]):

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{1}$$

$$\begin{aligned} \rho\frac{\mathrm{D}u}{\mathrm{D}t} &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\mu\frac{\partial u}{\partial x} + \tilde{\mu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] \\ &\quad + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right], \\ \rho\frac{\mathrm{D}v}{\mathrm{D}t} &= -\frac{\partial p}{\partial y} - \rho g + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] \\ &\quad + \frac{\partial}{\partial y}\left[2\mu\frac{\partial v}{\partial x} + \tilde{\mu}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right], \end{aligned} \tag{2}$$

$$\rho\frac{\mathrm{D}e}{\mathrm{D}t} - \frac{p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \operatorname{div}(\lambda \operatorname{grad} T) + \Phi; \tag{3}$$

ρ is the density, u and v are the velocity components in x- and y-direction, respectively, p is the pressure, e the specific internal energy, T the temperature, Φ the

†Private communication; see [16].

dissipation, λ the heat conductivity, μ the viscosity, and $\tilde{\mu}$ a second coefficient of viscosity, related to the bulk viscosity μ_b according to $\tilde{\mu} = \mu_b - \frac{2}{3}\mu$. Viscosity and thermal conductivity are assumed to be constant. The influence of variable transport coefficients has been investigated in [12].

The substantial time derivative is denoted by D/Dt, with

$$D/Dt = \partial/\partial t + u\,\partial/\partial x + v\,\partial/\partial y.$$

Without restrictions concerning the compressibility of the fluid, the system (1)–(3) is supplemented by the following relations between differential changes of state variables, where β is the thermal expansivity, K the isothermal compressibility, and c_v the specific heat capacity at constant volume [15]:

$$\mathrm{d}\rho = -\rho\beta\,\mathrm{d}T + \rho K\,\mathrm{d}p, \tag{4}$$

$$\mathrm{d}e = c_v\,\mathrm{d}T + \left(1 - \frac{\beta T}{Kp}\right)\frac{p}{\rho^2}\,\mathrm{d}\rho. \tag{5}$$

At the vertical wall periodic perturbations of the form

$$x = 0{:}\ \psi = \psi_0(y) + \Psi\,\mathrm{e}^{-i\omega t},\ \Psi = \text{const.}, \tag{6}$$

are prescribed, where ψ stands for any one of the dependent variables, and $\psi_0(y)$ describes the distribution of the variable ψ in the undisturbed basic state.

Small perturbations $\psi'(x,y,t)$ are introduced by the relation

$$\psi = \psi_0(y) + \psi'(x,y,t), \tag{7}$$

and after neglecting all quadratic terms the following set of linear equations is obtained from equations (1)–(3):

$$\frac{1}{\rho_0}\rho'_t + \frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}v' + u'_x + v'_y = 0, \tag{8}$$

$$u'_t = -\frac{1}{\rho_0}p'_x + \nu\Delta u' + (\nu+\tilde{\nu})(u'_x+v'_y)_x, \tag{9}$$

$$v'_t = -\frac{1}{\rho_0}p'_y + \nu\Delta v' + (\nu+\tilde{\nu})(u'_x+v'_y)_y - \frac{g}{\rho_0}\rho', \tag{10}$$

$$\rho_0 c_{v0}\left[T'_t - \frac{\beta_0 T_0}{K_0\rho_0^2 c_{v0}}\rho'_t + \left(\frac{\mathrm{d}T_0}{\mathrm{d}y} - \frac{\beta_0 T_0}{K_0\rho_0^2 c_{v0}}\frac{\mathrm{d}\rho_0}{\mathrm{d}y}\right)v'\right] = \lambda\Delta T'. \tag{11}$$

The subscripts x, y, t indicate partial derivatives. Note that the Boussinesq-approximation is not applied at the present stage since the term ρ'_t is retained in the continuity equation (8). In deriving equation (11) the internal energy was eliminated by means of (5). Another formulation of the energy equation, equivalent to (11), is the following:

$$\rho_0 c_{p0}\left[T' - \frac{\beta_0 T_0}{\rho_0 c_{p0}}p'_t + \gamma_0 v'\right] = \lambda\Delta T'. \tag{12}$$

where the coefficient γ_0 is the sum of the actual temperature gradient $\mathrm{d}T_0/\mathrm{d}y$ and the so-called "adiabatic temperature gradient" [10, 11]:

$$\gamma_0 = \frac{\mathrm{d}T_0}{\mathrm{d}y} + \frac{g\beta_0 T_0}{c_{p0}}. \tag{13}$$

Equation (12) can be obtained from equation (11) by using (4), the thermodynamic relationship [15]

$$c_{p0} - c_{v0} = \frac{\beta_0^2 T_0}{K_0\rho_0}, \tag{14}$$

and the hydrostatic equation

$$\frac{\mathrm{d}p_0}{\mathrm{d}y} = -\rho_0 g. \tag{15}$$

Using the thermodynamic equations [15]

$$\left(\frac{\partial s}{\partial T}\right)_p = \frac{c_p}{T}, \tag{16}$$

$$\left(\frac{\partial s}{\partial p}\right)_T = -\frac{\beta}{\rho}, \tag{17}$$

we can relate the coefficient γ_0 to the entropy gradient. The result is

$$\gamma_0 = \frac{T_0}{c_{p0}}\frac{\mathrm{d}s_0}{\mathrm{d}y}. \tag{18}$$

This relation, which will play an important role with regard to the stability of the basic state, is not obtained, if in the basic state, density changes with pressure are neglected [3] or the density gradient is put equal to zero [1].

As far as the disturbances (and not the basic states) are concerned the compressibility of the fluid can be neglected provided that we disregard sound waves and their (weak) interactions with the thermoconvective waves. Hence we shall use the approximation

$$\rho' = -\rho_0\beta_0 T', \tag{19}$$

in most parts of this paper, but we shall consider the coupling of thermoconvective waves and sound waves later in Section 5, thereby also providing a more rigorous justification of the approximation adopted here.

The perturbation equations (8)–(12) are linear but have, strictly speaking, variable coefficients $\rho_0(y)$, $T_0(y), \ldots$. Nevertheless we can approximate the coefficients by constant reference values provided that the vertical extension of the fluid layer to be considered is restricted such that the quantities ρ_0, $T_0, \ldots$ change but little across the layer. Note that this does not imply that the gradients $\mathrm{d}\rho_0/\mathrm{d}y$, $\mathrm{d}T_0/\mathrm{d}y, \ldots$, can be neglected.

3. ONE-DIMENSIONAL WAVE SOLUTIONS AND THE INSTABILITY OF THE UNDISTURBED STATE

If we assume that the fluid is unbounded in y-direction (see Fig. 1) and that the amplitude Ψ in the boundary condition (6) as well as the coefficients in the perturbation equations are constant, the solutions will be of the form

$$\psi = \psi_0(y)+\psi'(x,t). \tag{20}$$

As a consequence all partial derivatives with respect to y can be cancelled in the equations (8)–(12). Introducing the approximation (19) together with equation (20) into the equations (10) and (12) we obtain

$$v'_t - \nu v'_{xx} = g\beta_0 T', \tag{21}$$

$$T'_t - \kappa T'_{xx} = -\gamma_0 v'. \tag{22}$$

$\kappa = \lambda/\rho_0 c_{p_0}$ is the thermal diffusivity. In this case the momentum equation in y-direction, (21), and the energy equation, (22), are sufficient to describe thermoconvective waves. We want to find solutions of the system (21), (22) in the form of plane harmonic waves

$$\begin{Bmatrix} T' \\ v' \end{Bmatrix} = \begin{Bmatrix} \theta \\ V \end{Bmatrix} e^{i(kx-\omega t)}. \tag{23}$$

Introducing (23) into the system (21), (22) yields the dispersion relation

$$(-i\omega+\nu k^2)(-i\omega+\kappa k^2)+g\beta_0\gamma_0 = 0. \tag{24}$$

This is formally identical to the dispersion relation given by Luikov and Berkovsky [1] but with a different meaning of the parameter γ_0.

It is clear from equation (24) that a real wave number, i.e. vanishing damping, is obtained if (and only if) the wave frequency ω vanishes *and* the term $g\beta_0\gamma_0$, i.e. $\beta_0\gamma_0$, is negative. Since $g\beta_0\gamma_0$ is of the dimension of the square of a frequency, use of the following dimensionless variables is suggested:

$$\varepsilon = \frac{\omega}{\sqrt{-g\beta_0\gamma_0}}, \tag{25}$$

$$\bar{k} = kL, \quad \text{with } L = (-\kappa\nu/g\beta_0\gamma_0)^{1/4}. \tag{26}$$

This transforms the dispersion relation (24) into

$$(-i\varepsilon+\sqrt{Pr}\,\bar{k}^2)\left(-i\varepsilon+\frac{1}{\sqrt{Pr}}\bar{k}^2\right)-1 = 0, \tag{27}$$

where $Pr = \nu/\kappa$ is the Prandtl-number. Aiming at solutions for weakly damped waves, we expand for small values of ε according to

$$\bar{k} = \bar{k}_0+\varepsilon\bar{k}_1+\varepsilon^2\bar{k}_2+\ldots, \quad \text{with } \varepsilon \ll 1. \tag{28}$$

Introducing this expansion into the dispersion relation (27) leads to the following results:

$$\varepsilon^0: \quad \text{(I)} \quad \bar{k}_0 = 1, \qquad \text{(II)} \quad \bar{k}_0 = i, \tag{29}$$

$$\varepsilon^1: \quad \text{(I)} \quad \bar{k}_1 = i\frac{1}{4}\left(\sqrt{Pr}+\frac{1}{\sqrt{Pr}}\right), \qquad \text{(II)} \quad \bar{k}_1 = \frac{1}{4}\left(\sqrt{Pr}+\frac{1}{\sqrt{Pr}}\right), \tag{30}$$

$$\varepsilon^2: \quad \text{(I)} \quad \bar{k}_2 = \frac{1}{4}\left(1-\frac{1}{8}\left(\sqrt{Pr}+\frac{1}{\sqrt{Pr}}\right)^2\right), \qquad \text{(II)} \quad \bar{k}_2 = i\frac{1}{4}\left(1-\frac{1}{8}\left(\sqrt{Pr}+\frac{1}{\sqrt{Pr}}\right)^2\right). \tag{31}$$

The results describe a pair of thermoconvective waves, with the wave denoted by (I) being weakly damped, and the wave denoted by (II) being strongly damped. The damping coefficient is proportional to $(\sqrt{Pr}+1/\sqrt{Pr})$. This becomes a minimum at $Pr = 1$, which is therefore the point of optimal conditions for weakly damped thermoconvective waves. Taking into account the first and the second term of the expansion, we obtain for $Pr = 1$

$$\text{(I)} \quad k = \sqrt{\frac{\omega}{2\kappa}}\left[\frac{1}{\sqrt{\varepsilon/2}}+i\sqrt{\varepsilon/2}+\ldots\right], \qquad \text{(II)} \quad k = \sqrt{\frac{\omega}{2\kappa}}\left[\sqrt{\varepsilon/2}+i\frac{1}{\sqrt{\varepsilon/2}}+\ldots\right]. \tag{32}$$

This result is in remarkable contrast to the behaviour of classical thermal or shearing waves as was already discussed in Table 1.

The physical meaning of the perturbation parameter ε is seen more clearly when the expression $g\beta_0\gamma_0$, by using equations (4), (16)–(18), is rewritten as

$$g\beta_0\gamma_0 = \frac{g\beta_0 T_0}{c_{p_0}}\left(\frac{ds_0}{dy}\right) = -\left(\frac{g}{\rho_0}\frac{d\rho_0}{dy}+\frac{g^2}{c_0^2}\right), \tag{33}$$

where c_0 is the isentropic speed of sound in the undisturbed, basic state. If $\beta_0\gamma_0$ is positive the square root of the third expression in equation (33) is real and is known as the buoyancy frequency or Brunt–Väisälä frequency N [10, 11]. It is an important parameter in the theory of (stably) stratified fluids. With the present applications in mind it is very useful to extend the definition of the buoyancy frequency also to the case of negative values of $\beta_0\gamma_0$, therefore writing

$$\left.\begin{aligned} N^2 &= g\beta_0\gamma_0, \quad \text{if } \beta_0\gamma_0 > 0, \\ N^2 &= -g\beta_0\gamma_0, \quad \text{if } \beta_0\gamma_0 < 0. \end{aligned}\right\} \tag{34}$$

Comparing equation (34) with equation (25) we see that the parameter ε is the ratio between the wave frequency ω and the buoyancy frequency N:

$$\varepsilon = \omega/N, \quad \text{with } \beta_0\gamma_0 < 0. \tag{35}$$

The same frequency ratio, but for $\beta_0\gamma_0 > 0$, also appears as a characteristic parameter in the theory of internal gravity waves [10].

We indicated above that $\beta_0\gamma_0 < 0$ is a necessary condition for the existence of thermoconvective waves. With very few exceptions, the thermal expansivity β_0 of fluids is positive. For such "normal" fluids the necessary condition reduces to

$$\gamma_0 < 0 \tag{36}$$

or, in view of equation (18),

$$\frac{ds_0}{dy} < 0. \tag{37}$$

Since a vertically unbounded fluid is not mechanically stable at a negative entropy gradient (cf. e.g. [2], p. 10) the important conclusion is that in vertically unbounded normal fluids the necessary condition for thermoconvective waves contradicts the stability condition of the basic state. Although the condition (36) is already included in previous works [1, 3], the contradiction with the stability condition has been overlooked so far due to oversimplifications, mainly with regard to the compressibility of the fluid in the basic state.

In order to prevent instability of the basic state we can confine the fluid between two horizontal walls. This leads to the problem of two-dimensional wave propagation treated in the following section.

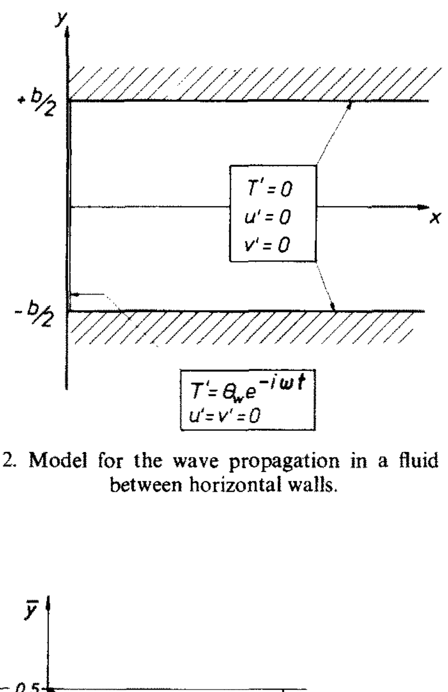

FIG. 2. Model for the wave propagation in a fluid layer between horizontal walls.

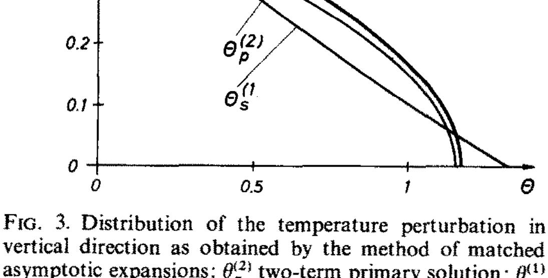

FIG. 3. Distribution of the temperature perturbation in vertical direction as obtained by the method of matched asymptotic expansions: $\theta_p^{(2)}$ two-term primary solution; $\theta_s^{(1)}$ one-term secondary solution; $\theta_+^{(2,1)}$ uniformly valid solution ($Ra = 1700$, $\delta^{1/2} = 0.155$).

4. THERMOCONVECTIVE WAVES IN A HORIZONTALLY BOUNDED FLUID LAYER

We now consider the propagation of thermoconvective waves in a fluid confined between nonmoving horizontal walls of constant temperature, as shown in Fig. 2. At subcritical Rayleigh-numbers the horizontal walls prevent the instability of the basic state, i.e. the onset of free convection. It is expected, however, that the horizontal walls give rise to an additional damping of the thermoconvective waves. The main question is now whether weakly damped thermoconvective waves are possible at all in a fluid layer bounded by horizontal walls.

4.1. *A two-parameter expansion of the eigenvalue problem*

The compressibility of the fluid is again neglected as far as perturbations are concerned, and it is recalled that the propagation of coupled thermoconvective waves and sound waves will be studied in Section 5.

The system of perturbation equations (8)–(11) together with the approximation (19) is now reduced to a single partial differential equation for the temperature perturbation. Using dimensionless variables

$$\left.\begin{aligned} \bar{x} = x/L, \quad \bar{y} = y/b, \quad \bar{t} = \omega t, \\ \bar{T} = T'/T_0(0), \end{aligned}\right\} \tag{38}$$

with the wave length of the weakly damped thermoconvective wave in the limit $\varepsilon \to 0$ as the characteristic length in x-direction, cf. equation (26), we obtain

$$\bar{D}_\nu\left\{\bar{D}_\kappa\left[\bar{\Delta}\bar{T} - \delta\frac{\tilde{N}^2 b}{g}\bar{T}_{\bar{y}}\right] - \varepsilon\delta\frac{N^2 b}{g}\bar{T}_{\bar{t}\bar{y}}\right\} - \bar{T}_{\bar{x}\bar{x}} = 0, \tag{39}$$

where

$$\left.\begin{aligned} &\bar{D}_\nu = \varepsilon\frac{\partial}{\partial\bar{t}} - \sqrt{Pr}\,\bar{\Delta}, \quad \bar{D}_\kappa = \varepsilon\frac{\partial}{\partial\bar{t}} - \frac{1}{\sqrt{Pr}}\bar{\Delta}, \\ &\bar{\Delta} = \frac{\partial^2}{\partial\bar{x}^2} + \delta\frac{\partial^2}{\partial\bar{y}^2}, \\ &\delta = (L/b)^2 = (-\kappa\nu/g\beta_0\gamma_0 b^4)^{1/2}. \end{aligned}\right\} \tag{40}$$

N^2 is given by equation (34), whereas $\tilde{N}^2$ is defined by

$$\tilde{N}^2 = -\frac{g}{\rho_0}\frac{d\rho_0}{dy}. \tag{41}$$

The differential equation (39) is supplemented by the following boundary conditions at the horizontal walls:

$$\begin{aligned} \bar{y} = \pm 1/2: \quad &\bar{T} = 0; \\ &v = 0, \text{ or } \bar{T}_{\bar{y}\bar{y}} = 0; \\ &u = 0, \text{ or } \bar{D}_\kappa\bar{T}_{\bar{y}} = 0. \end{aligned} \tag{42}$$

It should be mentioned here that the partial differential equation (39), being of sixth order with respect to $\bar{x}$

and $\bar{y}$, may yield not only two solutions for thermoconvective waves but also one solution for internal gravity waves. Contrary to thermoconvective waves, however, internal gravity waves can only exist at conditions of stable stratification, i.e. $\gamma_0 > 0$ or $ds_0/dy > 0$. The interesting possibility of a joint approach to thermoconvective waves and internal gravity waves has been discussed in more detail in [12]; the approach, however, is only suitable if either a two-dimensional formulation is adopted or compressibility effects are included, cf. also Section 5.

We now seek solutions for progressive waves of the form

$$\bar{T} = \theta(\bar{y})\,e^{i(\bar{k}\bar{x}-\bar{t})}. \tag{43}$$

Here the dimensionless complex wave number $\bar{k}$ is defined in the same way as for plane waves, cf. equation (26). By introducing equation (43) into the partial differential equation (39) we obtain the following ordinary differential equation for the amplitude θ ($'$ denotes the differentiation with respect to $\bar{y}$):

$$D_\nu\left\{D_\kappa\left[(\delta\theta''-\bar{k}^2\theta)-\delta\,\frac{\tilde{N}^2 b}{g}\,\theta'\right]+i\varepsilon\delta\,\frac{N^2 b}{g}\,\theta'\right\}+\bar{k}^2\theta = 0, \tag{44}$$

where

$$D_\nu = -i\varepsilon-\sqrt{Pr}\left(\delta\frac{d^2}{d\bar{y}^2}-\bar{k}^2\right),$$
$$D_\kappa = -i\varepsilon-\frac{1}{\sqrt{Pr}}\left(\delta\frac{d^2}{d\bar{y}^2}-\bar{k}^2\right) \tag{45}$$

The boundary conditions (42) yield

$$\theta = 0,\quad \theta'' = 0,\quad D_\nu\theta' = 0,\quad \text{at } \bar{y} = \pm 1/2. \tag{46}$$

Obviously we have to deal now with an eigenvalue problem with the solutions for the wave number $\bar{k}$ representing the eigenvalues.

The eigenvalue problem can be substantially simplified by considering the order of magnitude of the parameter δ, which characterizes the ratio between the height of the layer and the wave length of the weakly damped thermoconvective waves. By comparing the definition of δ, equation (40), and the definition of the Rayleigh-number

$$Ra = \frac{g\beta_0|\gamma_0|b^4}{\kappa\nu}, \tag{47}$$

we obtain (with $\gamma_0 < 0$)

$$\delta = 1/\sqrt{Ra}. \tag{48}$$

In a layer bounded by walls where no-slip conditions are prescribed, the critical Rayleigh-number is given by

$$Ra_{\text{crit}} = 1708, \tag{49}$$

see e.g. [9, p. 43]. As we are interested in weakly damped thermoconvective waves, we will choose the value of the Rayleigh-number as close to Ra_{crit} as possible in order to achieve large values of γ_0 and, according to equation (25), small values of ε. This has two consequences:

Firstly it follows from equation (13) that $|\gamma_0| \gg g\beta_0 T_0/c_{p_0}$, which leads to

$$\gamma_0 \approx \frac{dT_0}{dy},\quad \text{if } Ra \gg 1. \tag{50}$$

Hence the difference between γ_0 and dT_0/dy, although essential when we considered the stability of the unbounded fluid, can now be neglected, thereby bringing equation (47) *a posteriori* in full accord with the usual definition of the Rayleigh-number.

Secondly, with $\delta = O(1/\sqrt{Ra_{\text{crit}}}) \ll 1$, we conclude that δ, in addition to ε, can be regarded as a second perturbation parameter. In the limit $\delta \to 0$, equation (44) yields the dispersion relation (27) of the one-dimensional model. Collecting terms of order δ, however, results in a differential equation of the second order, whereas the full equation (44) is of sixth order. Therefore only two of the six boundary conditions (46) can be satisfied, and the regular expansion in terms of δ, with $\delta \to 0$, fails.

4.2. *Solution by the method of matched asymptotic expansions*

Avoiding possible misunderstanding of terms like "inner" and "outer", and following [13], we call the expansion, whose first-order term is determined without recourse to matching with another expansion, the "primary" expansion. In the present problem the primary expansion is obtained as $\delta \to 0$ with $\bar{y}$ fixed, therefore describing the behaviour of the wave in the bulk of the layer. In the boundary layers near the horizontal walls a secondary expansion, with a stretched vertical coordinate kept fixed, is appropriate.

Whereas the expansion in terms of δ is singular, the expansion in terms of ε can be expected to be regular as in the one-dimensional case. Thus the primary two-parameter expansion reads as follows:

$$\theta = \theta_{00}(\bar{y})+\varepsilon\theta_{10}(\bar{y})+\delta^{1/2}\theta_{01}(\bar{y})+O(\varepsilon^2,\varepsilon\delta^{1/2},\delta),$$
$$\bar{k} = \bar{k}_{00}+\varepsilon\bar{k}_{10}+\delta\bar{k}_{01}+\delta^{3/2}\bar{k}_{02}+O(\varepsilon^2,\varepsilon\delta,\delta^2), \tag{51}$$
$$(\varepsilon,\delta \to 0;\ \bar{y}\text{ fixed}).$$

Note that fractional powers of δ appear in order to allow the matching with the secondary solution.

As was already mentioned, the limit $\delta \to 0$ yields the dispersion relation of the one-dimensional (plane) waves. Thus

$$\bar{k}_{00} = \bar{k}_0,\quad \bar{k}_{10} = \bar{k}_1, \tag{52}$$

where $\bar{k}_0$ and $\bar{k}_1$ are given by equations (29) and (30). Similarly, θ_{10} is a "one-dimensional" correction to θ_{00}, and can be disregarded in the present analysis. Next, terms of order $\varepsilon^0\delta^1$ and $\varepsilon^0\delta^{3/2}$, respectively, are considered in equation (44). This yields

$$\theta_{00}''-\frac{\tilde{N}^2 b}{3g}\,\theta_{00}'-\tfrac{4}{3}\bar{k}_{01}\theta_{00} = 0, \tag{53}$$

and

$$\theta_{01}'' - \frac{\tilde{N}^2 b}{3g}\theta_{01}' - \tfrac{4}{3}\bar{k}_{01}\theta_{01} = \tfrac{4}{3}\bar{k}_{02}\theta_{00}. \tag{54}$$

Turning now to the secondary expansion, we introduce stretched coordinates near the upper and the lower wall, respectively:

$$\begin{aligned} Y &= (\bar{y}+\tfrac{1}{2})/\sqrt{\delta}, \quad \text{near } \bar{y} = -\tfrac{1}{2}; \\ Y &= (\tfrac{1}{2}-\bar{y})/\sqrt{\delta}, \quad \text{near } \bar{y} = +\tfrac{1}{2}. \end{aligned} \tag{55}$$

With the secondary expansion

$$\theta = \delta^{1/2}\vartheta_{00}(Y) + \varepsilon\vartheta_{10}(Y) + \ldots, \qquad (\varepsilon, \delta \to 0;\ Y \text{ fixed}), \tag{56}$$

we obtain from equations (44) and (46) the sixth-order differential equation

$$\left(\frac{d^2}{dY^2} - 1\right)^3 \vartheta_{00} + \vartheta_{00} = 0, \tag{57}$$

with the boundary conditions

$$\vartheta_{00} = 0,\ \vartheta_{00}'' = 0,\ \vartheta_{00}''' - \vartheta_{00}' = 0, \quad \text{at } Y = 0. \tag{58}$$

By a straightforward calculation, we now determine solutions of the secondary differential equation (57) which, on one hand, satisfy the boundary conditions (58), and, on the other hand, match asymptotically with the solution of the primary differential equation (53).

By use of Van Dyke's asymptotic matching principle (see e.g. [13], p. 206), the first-order result is

$$\theta_{00} = C\,e^{l\bar{y}}\cos(2M+1)\pi\bar{y}, \tag{59}$$

$$\vartheta_{00} = C_1 Y + C_2 + C_3 e^{-\alpha_1 Y} + C_4 e^{-\alpha_2 Y}, \tag{60}$$

$$\bar{k}_{01} = -\tfrac{3}{4}[l^2 + (2M+1)^2\pi^2], \tag{61}$$

where C is as a free constant, and

$$M = 0, 1, 2, \ldots, \tag{62a}$$

$$l = \tilde{N}^2 b/6g, \tag{62b}$$

$$\alpha_{1,2} = \left(\frac{\sqrt{3}}{2} \pm \frac{3}{4}\right)^{1/2} \pm i\left(\frac{\sqrt{3}}{2} \mp \frac{3}{4}\right)^{1/2}, \tag{62c}$$

$$C_1 = C(2M+1)\pi e^{\pm l/2}, \quad \text{at } \bar{y} = \pm\tfrac{1}{2}, \tag{62d}$$

$$C_2 = \frac{(\alpha_1^2/\alpha_2^2 - 1)C_1}{\alpha_1(1-\alpha_1^2)[1-\alpha_1(1-\alpha_2^2)/\alpha_2(1-\alpha_1^2)]} = -0.5373C_1, \tag{62e}$$

$$C_{3,4} = \frac{C_2}{\alpha_{1,2}^2/\alpha_{2,1}^2 - 1} = (0.2686 \pm 0.4653i)C_1. \tag{62f}$$

Note that due to the exponential term in equation (59) the wave amplitude is not symmetrical with respect to the centerline of the layer ($\bar{y} = 0$).

In order to keep the results for the second-order solution free of unessential details we now restrict the further analysis to the lowest mode ($M = 0$) of θ_{00} according to equation (59). This is justified if, for instance, the perturbations that are prescribed at the vertical wall $x = 0$ vary with $\bar{y}$ in the same way as does the lowest mode of θ_{00}. More complicated boundary conditions at the vertical wall would require a superposition of higher-order modes. Solving the second-order equation (54) with $M = 0$ in θ_{00}, and applying again the asymptotic matching condition to determine constants of integration, we finally obtain

$$\begin{aligned} \theta_{01} &= e^{l\bar{y}}[B\cos\pi\bar{y} + 2\pi C(C_2/C_1)\bar{y}\sin\pi\bar{y}], \\ \bar{k}_{02} &= 3\pi^2 C_2/C_1 = -1.6119\pi^2. \end{aligned} \tag{63}$$

Similar to C, also the constant B remains undetermined within the eigenvalue problem. Both constants are available for satisfying boundary conditions at the vertical wall ($x = 0$).

Summarizing the results for the complex wave number, we obtain for the lowest mode of the weakly damped thermoconvective wave the expression

$$\bar{k} = \bar{k}_{00} + \varepsilon\bar{k}_{10} + \delta\bar{k}_{01} + \delta^{3/2}\bar{k}_{02} + \ldots \tag{64}$$

with $\bar{k}_{00} = 1$,

$$\begin{aligned} \bar{k}_{10} &= i\tfrac{1}{4}(\sqrt{Pr} + 1/\sqrt{Pr}), \\ \bar{k}_{01} &= -\tfrac{3}{4}[(\tilde{N}^2 b/6g)^2 + \pi^2], \\ \bar{k}_{02} &= -15.909. \end{aligned} \tag{65}$$

It is seen from these results that at large Rayleigh-numbers the effect of horizontal walls on the wave number is of order $\delta = Ra^{-1/2}$ only. This effect is due to the development of an amplitude profile between the walls. On the other hand, due to the no-slip condition at the walls a boundary layer forms, giving rise to a correction term of order $\delta^{3/2}$ in the wave number. It must be emphasized that both correction terms are real and negative, which means that the real part of the wave number is reduced, and the wave length as well as the phase velocity is increased. Within this order of magnitude, i.e. terms $O(\delta^{3/2})$ included, the damping coefficient is unaffected by the presence of the horizontal walls. An additional damping due to the friction at the walls can only appear in terms of higher order compared to $\delta^{3/2}$.

Thus it has been shown that in a horizontal fluid layer, bounded by rigid walls, weakly damped thermoconvective waves are possible if the value of the Rayleigh-number is close to the (large) value of the critical Rayleigh-number. Furthermore, the results of the one-dimensional model can be taken as a first approximation in the two-dimensional case as far as the wave number is concerned. This is of course not true for the amplitude which strongly depends on the lateral coordinate in the two-dimensional case.

It seems appropriate at this stage to note that the Boussinesq-approximation, although often used in natural convection problems, has not been applied in our analysis so far. Applying the Boussinesq-approximation would result in neglecting the first and the second term in the continuity equation (8). The first term leads to the term with the coefficient $\varepsilon\delta N^2 b/g$ in equation (39), and it eventually becomes a negligible higher-order term in our expansion. The second term appears in equation (39) as the term with the coefficient $\delta\tilde{N}^2 b/g$; this term is responsible for the asymmetry of the amplitude, cf. equations (59) and (62b), and it also gives rise to a modification of the wave number, cf. equation (65). Although these effects are qualitatively interesting, they are not very important from a quantitative point of view since

$$\tilde{N}^2 b/g \ll 1, \tag{66}$$

under many conditions of practical interest. Two examples are given in Table 2. The values in the first example are based on data of experiments [7]. The second example is intended to show that a rather modest increase in the layer height has quite a large effect on the important parameters, especially the temperature difference between the walls and the wave period.

Table 2.

Rayleigh-number		$Ra = 1700$	
Perturbation parameter		$\delta = 0.024$	
Boundary-layer thickness related to the depth of the layer		$\sqrt{\delta} = 0.155$	
Medium: Air at 300 K			
Height of the layer b	[m]	1.17×10^{-2}	8.45×10^{-2}
Buoyancy frequency N	[s^{-1}]	5.22	0.1
Period $\frac{2\pi}{\omega}$ at $\varepsilon = 0.1$	[s]	12	628
Parameter γ_0	[$K\,m^{-1}$]	8.3×10^2	0.31
$\frac{g\beta_0 T_0(0)}{c_{p_0}}$	[$K\,m^{-1}$]	0.98×10^{-2}	0.98×10^{-2}
Temperature difference $T_0(-1/2) - T_0(+1/2)$	[K]	9.7	0.026
$\frac{\tilde{N}^2 b}{g}$	[1]	-3.25×10^{-2}	-0.79×10^{-4}

In order to find a uniformly valid solution for the amplitude, we add the primary and the secondary solutions and subtract their common part (additive composition, cf. e.g. [13, p. 208]). Adopting the Boussinesq-approximation, we obtain from equations (59), (60), and (63)

$$\theta_+^{(2,1)} = (1+\sqrt{\delta})\cos\pi\bar{y} + \sqrt{\delta}\,2\pi(C_2/C_1)\bar{y}\sin\pi\bar{y} + \sqrt{\delta}[C_3 e^{-\alpha_1 Y} + C_4 e^{-\alpha_2 Y}]. \tag{67}$$

The superscript (2, 1) indicates that the solution is accurate to second order in the primary layer (bulk layer), and to first order in the secondary layers (boundary layers).

In Fig. 3 this solution is shown together with the second-order primary and the first-order secondary solution for the upper half of the layer. As the Rayleigh-number is limited by the critical value, the value of $\delta^{1/2}$ cannot be chosen smaller than 0.155. Hence the depth of the boundary layer as well as the discrepancy between the uniformly valid solution and the secondary solution are relatively large. The two-term primary solution shows very clearly the displacement effect of the boundary layer.

5. THE INTERACTION BETWEEN THERMOCONVECTIVE WAVES AND SOUND WAVES

In general, the thermal and transverse oscillations of thermoconvective waves are coupled with perturbations of the density, the pressure and the longitudinal component (x-components) of the velocity, cf. equations (8)–(11). Thus, with the compressibility of the fluid fully taken into account, a sound wave will propagate simultaneously with the thermoconvective waves. According to the results of the previous section it is justified to study the influence of the sound wave again by means of a one-dimensional model, although we have to be aware of the modifications due to the horizontal boundaries of the layer.

We introduce now the one-dimensional formulation (20) into the system of equations (8)–(11), supplemented by the general relations (4) and (5) without any restrictions concerning the compressibility of the fluid. Assuming plane harmonic waves as indicated by equation (23), one can derive the following dispersion relation:

$$\bar{k}^6\{1-i\varepsilon\eta A_1\}-\bar{k}^4\left\{i\varepsilon\left[\sqrt{Pr}\left(1-i\varepsilon\eta A_1\frac{c_{v0}}{c_{p0}}\right)+\frac{1}{\sqrt{Pr}}(1-i\varepsilon\eta A_2)\right]+\frac{\eta}{\sqrt{Pr}}A_1B\right\}-\bar{k}^2\{1-i\varepsilon\eta A_3B+\varepsilon^2(1-i\varepsilon\eta)A_4\}+\varepsilon^2\eta(B+\varepsilon^2)=0. \quad (68)$$

The dimensionless wave number $\bar{k}$ and the parameter ε have already been defined by equations (26) and (25), respectively. The new parameter η is given by

$$\eta = N\sqrt{\kappa\nu}/c_0^2, \quad (69)$$

with c_0 as the isentropic speed of sound in the undisturbed state, and N as the buoyancy frequency, cf. equation (34). Furthermore the following abbreviations are introduced:

$$\begin{aligned}
A_0 &= c_{p_0}/c_{v_0}\sqrt{Pr},\\
A_1 &= A_0 Pr\hat{\nu}/\nu,\\
A_2 &= A_0 Pr(1+\hat{\nu}/\nu),\\
A_3 &= A_0+\sqrt{Pr}(1+\hat{\nu}/\nu),\\
A_4 &= A_0+\sqrt{Pr}\hat{\nu}/\nu,\\
B &= -\tilde{N}^2/N^2,\\
\hat{\nu} &= (2\mu+\tilde{\mu})/\rho_0 \equiv (\tfrac{4}{3}\mu+\mu_b)/\rho_0.
\end{aligned} \quad (70)$$

The physical meaning of the parameter η becomes immediately clear if we regard the phase velocities of the two thermoconvective waves (I) and (II) at small values of the parameter ε, cf. equations (26), (28), and (29):

$$\begin{aligned}
\left(\frac{\omega}{k_r}\right)_{\mathrm{I}} &\sim \varepsilon[N\sqrt{\kappa\nu}]^{1/2},\\
\left(\frac{\omega}{k_r}\right)_{\mathrm{II}} &\sim [N\sqrt{\kappa\nu}]^{1/2}.
\end{aligned} \quad (71)$$

Hence the parameter η is the square of the ratio between the phase velocity of the strongly damped thermoconvective wave (wave II) and the isentropic speed of sound. Although the larger of the two phase velocities $(\omega/k_r)_{\mathrm{I}}$ and $(\omega/k_r)_{\mathrm{II}}$ is related to c_0, η can be treated as a small perturbation parameter. There is an upper bound for N since the critical Rayleigh-number must not be exceeded (see Section 4), and at the typical value $N = 1\,\mathrm{s}^{-1}$ we obtain for air at 300 K

$$\eta = 1.7\times10^{-10}.$$

Asymptotic solutions for the wave numbers of the thermoconvective waves can be found by introducing the two-parameter expansion

$$\bar{k} = \bar{k}_{00}+\varepsilon\bar{k}_{01}+\eta\bar{k}_{10}+O(\varepsilon^2, \varepsilon\eta, \eta^2), \quad (72)$$

into equation (68). The solutions for $\bar{k}_{00}$ and $\bar{k}_{01}$ are identical to the values of $\bar{k}_0$ and $\bar{k}_1$ in equations (29) and (30), that are the results obtained with the compressibility neglected in the disturbances. For the first-order correction due to compressibility we obtain

$$\begin{aligned}
&(\mathrm{I}) \quad \bar{k}_{10} = A_1B/4\sqrt{Pr},\\
&(\mathrm{II}) \quad \bar{k}_{10} = -iA_1B/4\sqrt{Pr}.
\end{aligned} \quad (73)$$

If the propagation of thermoconvective waves is considered the basic state is typically such that the constants A_1 and B, cf. equation (70), are positive. Hence the real wave number of the weakly damped thermoconvective wave (wave I) is increased and the damping of the strongly damped thermoconvective wave (wave II) is decreased. This leads to a reduced attenuation per wave length in both cases.

Since the dimensionless wave number $\bar{k}$ was defined by using the wave length of thermoconvective waves as a reference length, the expansion (72) cannot yield solutions for the sound wave. If we want to keep the definition of $\bar{k}$ unchanged, we can obtain the solution for the sound wave by changing the expansion to

$$\bar{k} = \varepsilon\eta^{1/2}[\bar{k}_{00}+\varepsilon\bar{k}_{01}+\eta k_{10}+\varepsilon^2\bar{k}_{02}+\varepsilon\eta\bar{k}_{11}+\eta^2\bar{k}_{20}+\ldots]. \quad (74)$$

The classical relation for the isentropic sound wave, $k = \omega/c_0$, would correspond to $\bar{k} = \varepsilon\eta^{1/2}$. Introducing equation (74) into the dispersion relation (68) gives

$$\begin{aligned}
&\bar{k}_{00} = \sqrt{B}, \quad \bar{k}_{01} = \bar{k}_{10} = 0,\\
&\bar{k}_{02} = (1-B)/2\sqrt{B}, \quad \bar{k}_{11} = i(A_3/2)B^{3/2}.
\end{aligned} \quad (75)$$

Surprisingly already the lowest order of the expansion of the wave number deviates from the classical result. Though B is approximately equal to one at conditions that are typical of thermoconvective waves, it could adopt any other value in a general case; even $B = 0$ or $B < 0$ would be possible. An explanation for this phenomenon is found by realizing that in a stratified fluid sound waves may be coupled with internal gravity waves. This problem has been treated in [14] for the case of horizontally propagating plane waves, and the above results have been verified. Further details can also be found in [12]. We just note here that by extending the one-dimensional treatment from the incompressible to the compressible case, we are retaining possible solutions for internal gravity waves.

Now turning our attention to the wave amplitudes, we sum up the contributions of the three simultaneously propagating waves,

$$\begin{aligned}
T' &= \sum_{j=1}^{\mathrm{III}} \theta_j e^{i(k_jx-\omega t)},\\
v' &= \sum_{j=1}^{\mathrm{III}} V_j e^{i(k_jx-\omega t)},\\
u' &= \sum_{j=1}^{\mathrm{III}} U_j e^{i(k_jx-\omega t)},
\end{aligned} \quad (76)$$

to satisfy the boundary conditions at the vertical wall

$$\sum_{j=1}^{\mathrm{III}} \theta_j = \theta_0, \quad \sum_{j=1}^{\mathrm{III}} V_j = V_0, \quad \sum_{j=1}^{\mathrm{III}} U_j = U_0. \quad (77)$$

Subscripts $j = \mathrm{I}$ and $j = \mathrm{II}$ refer to the two thermoconvective waves, $j = \mathrm{III}$ to the sound wave. Boundary values at the wall are denoted by the subscript 0.

As our aim is to compare the magnitudes of the amplitudes, we shall leave possible phase angles between θ_0, V_0 and U_0 out of consideration. By eliminating the amplitudes V_j and U_j with the use of the governing one-dimensional equations, we obtain from the equations (77) a system of three linear algebraic equations for the three unknown amplitudes θ_j. To compute the coefficients of these equations we use the asymptotic solutions for the wave numbers according to equations (72) and (74). Furthermore we define the following dimensionless amplitudes:

$$\bar{\theta}_j = \theta_j \beta_0; \quad \bar{V}_j = V_j \frac{N}{g} \sqrt{Pr};$$
$$\bar{U}_j = \frac{U_j}{c_0} A_0 \sqrt{Pr}/\sqrt{B}, \quad j = 0, \mathrm{I}, \mathrm{II}, \mathrm{III}. \tag{78}$$

By considering only the leading terms of the expansions and solving the system of linear algebraic equations, we finally obtain the temperature amplitudes $\bar{\theta}_j$ in terms of the boundary values $\bar{\theta}_0$, $\bar{V}_0$, $\bar{U}_0$:

$$\begin{Bmatrix} \bar{\theta}_{\mathrm{I}} \\ \bar{\theta}_{\mathrm{II}} \\ \bar{\theta}_{\mathrm{III}} \end{Bmatrix} = \begin{Bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 A_0 \sqrt{B}(1-i)\eta^{1/2} & -1/2 A_0 \sqrt{B}(1+i)\eta^{1/2} & -1 \end{Bmatrix} \begin{Bmatrix} \bar{\theta}_0 \\ \bar{V}_0 \\ \bar{U}_0 \end{Bmatrix}. \tag{79}$$

Using again the governing equations that were used to eliminate the velocity components, one can derive from equation (79) the following solutions for the amplitudes $\bar{V}_j$ and $\bar{U}_j$:

$$\begin{Bmatrix} \bar{V}_{\mathrm{I}} \\ \bar{V}_{\mathrm{II}} \\ \bar{V}_{\mathrm{III}} \end{Bmatrix} = \begin{Bmatrix} 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ -\frac{1+i}{2}\sqrt{B}\,\varepsilon\eta^{1/2} & -\frac{1+i}{2}\sqrt{B}\,\varepsilon\eta^{1/2} & \frac{i}{A_0}\varepsilon \end{Bmatrix} \begin{Bmatrix} \bar{\theta}_0 \\ \bar{V}_0 \\ \bar{U}_0 \end{Bmatrix}, \tag{80}$$

$$\begin{Bmatrix} \bar{U}_{\mathrm{I}} \\ \bar{U}_{\mathrm{II}} \\ \bar{U}_{\mathrm{III}} \end{Bmatrix} = \begin{Bmatrix} -\frac{i}{2}A_0\sqrt{B}\eta^{1/2} & -\frac{i}{2}A_0\sqrt{B}\eta^{1/2} & -\frac{i}{2}A_0\sqrt{B}\eta^{1/2} \\ \frac{1}{2}A_0\sqrt{B}\eta^{1/2} & -\frac{1}{2}A_0\sqrt{B}\eta^{1/2} & \frac{1}{2}A_0\sqrt{B}\eta^{1/2} \\ -\frac{1-i}{2}A_0\sqrt{B}\eta^{1/2} & \frac{1+i}{2}A_0\sqrt{B}\eta^{1/2} & 1 \end{Bmatrix} \begin{Bmatrix} \bar{\theta}_0 \\ \bar{V}_0 \\ \bar{U}_0 \end{Bmatrix}$$

For details of the calculations the reader is referred to [12].

As far as orders of magnitude are concerned, the results obtained for the amplitudes can be summarized as follows. If the waves are induced by a horizontal motion of the (vertical) wall, the dimensionless temperature amplitudes of the sound wave and of the thermoconvective waves are all of the same order as the dimensionless amplitude of the wall motion (i.e. $\bar{U}_0$). In this case the contributions of the sound wave are essential and cannot be neglected. If, on the other hand, the waves are due to temperature oscillations at the wall and/or a vertical motion of the wall, the temperature and velocity amplitudes of the sound wave are small of the order $O(\eta^{1/2})$ and $O(\varepsilon\eta^{1/2})$, respectively, in comparison to the wall values, and can therefore be neglected under most circumstances.

However, one has to pay attention to the fact that the attenuation of the sound wave is much smaller than the attenuation of even the weakly damped thermoconvective wave. While the damping of the thermoconvective wave I is of the order $O(\varepsilon)$, the damping of the sound wave is only of the order $O(\varepsilon^2\eta^{3/2})$, cf. equations (74) and (75).

6. SUMMARY

Thermoconvective waves are strongly coupled thermal and shearing waves in a stratified fluid, with the coupling being due to buoyancy forces. In contrast to ordinary thermal waves and ordinary shearing waves, thermoconvective waves can be weakly damped.

Avoiding any assumption about the compressibility of the fluid in the undisturbed basic state it has been shown in Section 3 that the conditions for the propagation of one-dimensional thermoconvective waves contradict the stability criterion for a vertically unbounded fluid. This contradiction was overlooked by previous authors [1, 3] due to various assumptions and approximations concerning the basic state of the fluid.

In order to prevent instability of the basic state the fluid can be confined between two horizontal walls. This leads to the problem of two-dimensional wave propagation treated in Section 4.

To find asymptotic solutions of the eigenvalue problem a two-parameter expansion is carried through. The first perturbation parameter, denoted by ε, is the ratio of the wave frequency ω and the buoyancy frequency N; only if ε is very small are weakly damped thermoconvective waves possible at all. The second perturbation parameter, δ, is the square of the ratio of wave length and height of the layer. The parameter δ is related to the Rayleigh-number by $\delta = Ra^{-1/2}$; it can be regarded as small compared to 1 at values of Ra close to the critical Rayleigh-number of ~ 1700.

Whereas the expansion in terms of ε is regular, the expansion in terms of δ leads to a singular perturbation problem that has been solved by the method of matched asymptotic expansions. The first correction to the results of the one-dimensional model consists of an increase of the wavelength of the order $O(\delta)$ and is due to the formation of an amplitude profile between the horizontal walls. The influence of the boundary layer solution on the wave number is of the order $O(\delta^{3/2})$. It is very interesting that this modification, too, does not yield an additional damping of the waves but further increases the wave length. This can be interpreted as a displacement effect of the boundary layer. As the damping due to the friction at the horizontal walls is shown to be small of higher order, it follows that the horizontal walls do not prevent the existence of weakly damped thermoconvective waves provided that the Rayleigh-number is large.

Taking into account the compressibility of the fluid not only as far as the basic state is concerned but also with respect to the disturbances, the interaction between thermoconvective waves and sound waves has been investigated in Section 5. Regarding the thermoconvective waves it has been shown that the compressibility effects slightly reduce the attenuation per wave length; the reduction is of the order of the very small parameter η, where η is the square of the ratio of a thermoconvective phase velocity and the speed of sound. On the other hand, the simultaneously propagating sound wave is considerably modified. This modification is connected with the propagation of internal gravity waves in stratified fluids.

Finally, the two thermoconvective wave modes and the sound wave have been superposed to satisfy the boundary conditions at the vertical wall, and the amplitudes of the three partial waves have been determined. The results indicate that the contribution of the sound wave can be neglected if we are primarily interested in the propagation of thermal and transversal (shearing) oscillations, but that it must be taken into account when at the vertical wall also longitudinal oscillations are prescribed.

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REFERENCES

1. A. V. Luikov and B. M. Berkovsky, Thermoconvective waves, *Int. J. Heat Mass Transfer* **13,** 741 (1970). cf. also *Dokl. Akad. Nauk BSSR* **13,** 316 (1969).
2. L. D. Landau and E. M. Lifschitz, *Lehrbuch der Theoretischen Physik. Bd. VI: Hydrodynamik.* Akademie, Berlin (1971).
3. B. M. Berkovsky *et al.*, Free convection. In *Progress in Heat and Mass Transfer*, Vol. 4, edited by O. G. Martynenko *et al.*, pp. 191–233. Pergamon Press, Oxford (1971).
4. A. V. Luikov and B. M. Berkovsky, Thermoconvective waves in viscoelastic liquids, *Inzh.-Fiz. Zh.* **16**, No. 5 (1969).
5. B. M. Berkovsky and A. K. Sinitsyn, Thermoconvective waves in a layer with free boundaries, *Inzh.-Fiz. Zh.* **25**, No. 1 (1974).
6. B. M. Berkovsky and A. K. Sinitsyn, Thermoconvective waves in a horizontal cavity, *Fluid Dynamics* **10**, 174 (1975). Transl. from *Izv. Akad. Nauk SSSR, Mekhanika Zhidkosti i Gaza*, No. 1, 180 (1975).
7. Yu. I. Barkov, B. M. Berkovsky and V. E. Fertman, Experimental study of thermoconvective waves, *Fluid Dynamics* **10**, 187 (1975). Transl. from *Izv. Akad. Nauk SSSR, Mekhanika Zhidkosti i Gaza*, No. 2, 187–189 (1975).
8. K. Oswatitsch, *Physikalische Grundlagen der Strömungsmechanik, Handbuch der Physik*, Hrsg. S. Flügge; Band VIII/1. Springer, Berlin (1959).
9. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press, Oxford (1961).
10. J. S. Turner, *Buoyancy Effects in Fluids*, Cambridge University Press, Cambridge (1973).
11. E. E. Gossard and W. H. Hooke, *Waves in the Atmosphere*, Elsevier, Amsterdam (1975).
12. H. Keck, Thermokonvektive Wellen unter Berücksichtigung von Kompressibilität, Wandeinflüssen und Wärmestrahlung, Dissertation Techn. Univ., Wien (1977).
13. W. Schneider, *Mathematische Methoden der Strömungsmeckanik*. Vieweg, Braunschweig (1978).
14. H. Keck, Über die eindimensionale Ausbreitung kleiner Störungen in einem kompressiblen, geschichteten Medium im Schwerefeld, *Z. Flugwiss. Weltraumforsch.* **1,** 42 (1977).
15. E. Becker, *Gasdynamik*, p. 25. Teubner, Stuttgart (1966).
16. B. M. Berkovsky, V. E. Fertman, A. K. Sinitsyn and Yu. I. Barkov, A theoretical and experimental study of thermal disturbances propagating in a fluid layer heated from below, *J. Fluid Mech.* To be published.

ONDES THERMOCONVECTIVES DANS UN FLUIDE COMPRESSIBLE

Résumé—Les conditions de la propagation des ondes thermoconvectives monodimensionnelles sont en contradiction avec le critère de stabilité pour un fluide verticalement illimité. Ceci conduit au problème de la propagation d'une onde bidimensionnelle dans une couche de fluide stratifiée, confinée entre les parois horizontales. Le problème de valeurs propres est résolu par un développement à deux paramètres qui est singulier par rapport à l'un des paramètres. Les résultats montrent que les parois horizontales ne suppriment pas l'existence d'ondes thermoconvectives faiblement amorties lorsque le nombre de Rayleigh est grand. Finalement, l'intéraction des ondes thermoconvectives et des ondes sonores est étudiée et les amplitudes des différents modes sont déterminées.

THERMOKONVEKTIVE WELLEN IN EINEM KOMPRESSIBLEN FLUID

Zusammenfassung—Es wird gezeigt, daß die Bedingungen für die Ausbreitung von eindimensionalen thermokonvektiven Wellen dem Stabilitätskriterium für ein vertikal unbegrenztes Fluid widersprechen. Das führt zum Problem der zweidimensionalen Wellenausbreitung in einem geschichteten Fluid, das zwischen horizontalen Wänden eingeschlossen ist. Das Eigenwert-Problem wird mittels einer Zwei-Parameter-Entwicklung gelöst, die bezüglich eines der beiden Parameter singulär ist. Die Ergebnisse zeigen, daß die horizontalen Wände die Existenz schwach gedämpfter thermokonvektiver Wellen nicht verhindern, vorausgesetzt daß die Rayleigh-Zahl sehr groß ist. Schließlich wird die Wechselwirkung zwischen thermokonvektiven Wellen und Schallwellen untersucht, und die Amplituden der einzelnen Wellen werden bestimmt.

ТЕРМОКОНВЕКТИВНЫЕ ВОЛНЫ В СЖИМАЕМОЙ ЖИДКОСТИ

Аннотация — Показано, что условия распространения одномерных термоконвективных волн противоречат критерию устойчивости вертикально неограниченной жидкости. Это приводит к проблеме двухмерного распространения волн в стратифицированном слое жидкости, заключенном между горизонтальными стенками. Задача на собственные значения решается с помощью двухпараметрического разложения, являющегося сингулярным по отношению к одному из параметров. Результаты показывают, что при наличии горизонтальных стенок также имеют место слабо затухающие термоконвективные волны, но только в случае большого значения числа Релея. Наконец, рассматривается взаимодействие термоконвективных и звуковых волн и определяются амплитуды различных волновых мод.